Restricted Lucas congruences for Apéry numbers modulo p^2

Eric Rowland
Hofstra University

Joint work with Reem Yassawi

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Apéry numbers

$A(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ arose in Apéry's proof that $\zeta(3)$ is irrational.

$A(n)_{n \geq 0}$: $1, 5, 73, 1445, 33001, 819005, 21460825, \ldots$

Theorem (Gessel 1982)

Let p be a prime. The Apéry numbers satisfy the Lucas congruence

$$A(n) \equiv A(n_0)A(n_1) \cdots A(n_\ell) \mod p,$$

where $n_\ell \cdots n_1 n_0$ is the standard base-p representation of n.

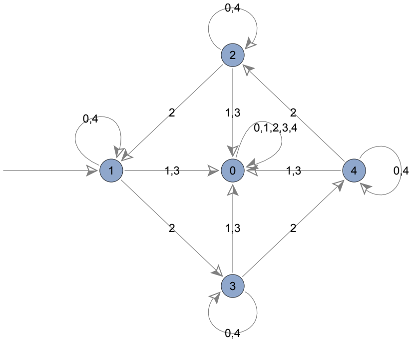
Example

Let $p = 5$ and $n = 447 = 3242_5$.

$$A(447) \equiv A(2)A(4)A(2)A(3) \equiv 3 \cdot 1 \cdot 3 \cdot 0 \equiv 0 \mod 5.$$

$(A(n) \bmod 5)_{n \geq 0}$ is 5-automatic.

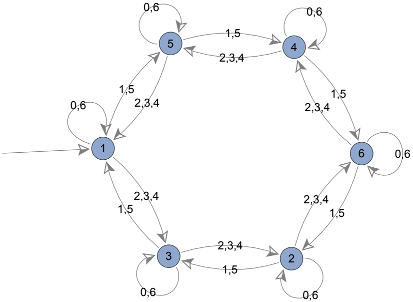
Create one state for each residue modulo 5.
Edges encode transitions between states.



$$A(447) \equiv A(2)A(4)A(2)A(3) \equiv 3 \cdot 1 \cdot 3 \cdot 0 \equiv 0 \mod 5$$

Mod 7

$A(d) \not\equiv 0 \mod 7$ for all $d \in \{0, 1, \ldots, 6\}$.



Therefore $A(n) \not\equiv 0 \mod 7$ for all $n \geq 0$.

Diagonals of rational power series

The diagonal of a formal power series is

$$\operatorname{diag} \sum_{n_1, n_2, \ldots, n_k \geq 0} a_{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} := \sum_{n \geq 0} a_{n,n,\ldots,n} x^n.$$

Straub (2014):

$$\operatorname{diag} \frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4} = \sum_{n \geq 0} A(n) x^n.$$

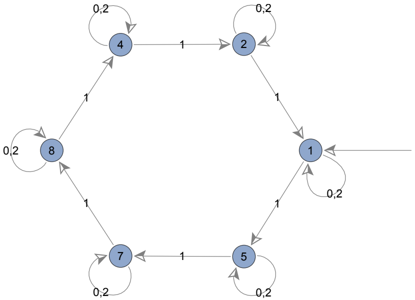
Theorem (Denef–Lipshitz 1987)

Let $\alpha \geq 1$. Let $R(\mathbf{x}), Q(\mathbf{x}) \in \mathbb{Z}_p[\mathbf{x}]$ such that $Q(0, \ldots, 0) \not\equiv 0 \mod p$. Then the coefficient sequence of $\left(\operatorname{diag} \frac{R(\mathbf{x})}{Q(\mathbf{x})}\right) \mod p^\alpha$ is p-automatic.

$\mathbb{Z}_p$ denotes the set of p-adic integers.

Therefore $(A(n) \mod p^\alpha)_{n \geq 0}$ is p-automatic for every prime power p^α.

Mod 9

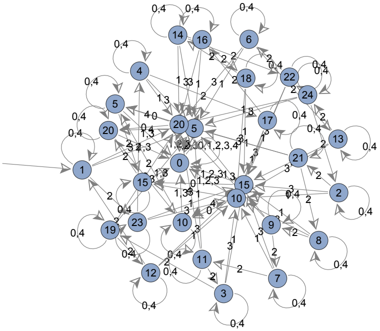


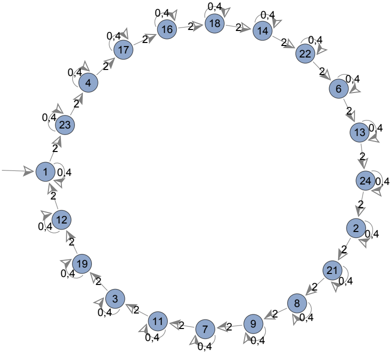
Theorem (Gessel 1982)

$$A(n) \equiv A(n_0)A(n_1)\cdots A(n_\ell) \mod 9,$$

where $n_\ell \cdots n_1 n_0$ is the base-3 representation of n.

For $p \geq 5$, the Lucas congruence does not hold modulo p^2.





Restrict the digit set.

Theorem (Rowland–Yassawi 2015)

If each digit in the base-5 representation of n belongs to $\{0, 2, 4\}$*, then* $A(n) \equiv A(n_0)A(n_1)\cdots A(n_\ell) \mod 25$*.*

Search for digit sets

Which digit sets support Lucas congruences for $A(n) \bmod p^2$?

For large p, the automaton for $A(n) \bmod p^2$ is hard to compute.

Experimental approach: Test all 3-element subsets of $\{0, 1, \ldots, p-1\}$.

p	digits sets
3	$\{0, 1, 2\}$
5	$\{0, 1, 3\}, \{0, 2, 4\}$
7	$\{0, 2, 3, 4, 6\}$
11	$\{0, 5, 10\}$
13	$\{0, 6, 12\}$
17	$\{0, 3, 13\}, \{0, 8, 16\}$
19	$\{0, 8, 10\}, \{0, 9, 18\}$

Symmetry

Theorem (Malik–Straub 2016)

$A(d) \equiv A(p-1-d) \mod p$ *for each* $d \in \{0, 1, \ldots, p-1\}$.

Let $D(p) := \left\{ d \in \{0, 1, \ldots, p-1\} : A(d) \equiv A(p-1-d) \mod p^2 \right\}$.
In particular, $\{0, \frac{p-1}{2}, p-1\} \subseteq D(p)$. $\{0, 2, 4\} \subseteq D(5)$

Theorem (Rowland–Yassawi)

The digit set $D(p)$ supports a restricted Lucas congruence for the Apéry numbers modulo p^2.

That is, if each base-p digit of n belongs to $D(p)$, then

$$A(n) \equiv A(n_0)A(n_1) \cdots A(n_\ell) \mod p^2,$$

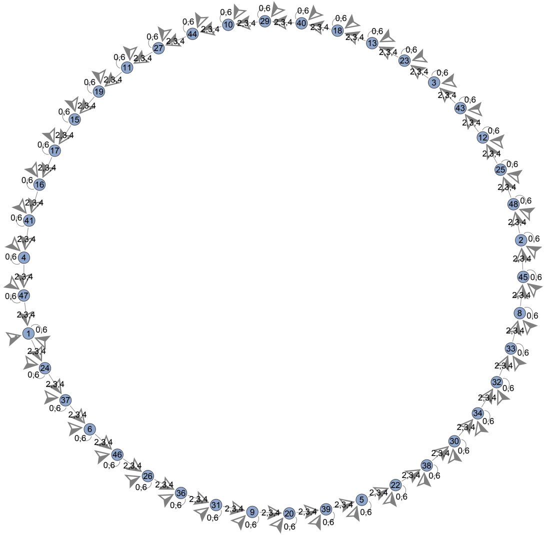
where $n_\ell \cdots n_1 n_0$ is the standard base-p representation of n.

Large digit sets

Primes p with $|D(p)| \geq 4$:

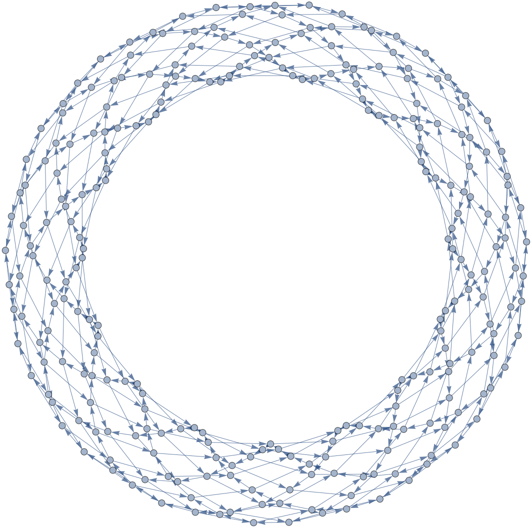
p	$D(p)$
7	$\{0, 2, 3, 4, 6\}$
23	$\{0, 7, 11, 15, 22\}$
43	$\{0, 5, 18, 21, 24, 37, 42\}$
59	$\{0, 6, 29, 52, 58\}$
79	$\{0, 18, 39, 60, 78\}$
103	$\{0, 17, 51, 85, 102\}$
107	$\{0, 14, 21, 47, 53, 59, 85, 92, 106\}$
127	$\{0, 17, 63, 109, 126\}$
131	$\{0, 62, 65, 68, 130\}$
139	$\{0, 68, 69, 70, 138\}$
151	$\{0, 19, 75, 131, 150\}$
167	$\{0, 35, 64, 83, 102, 131, 166\}$

Mod 7^2



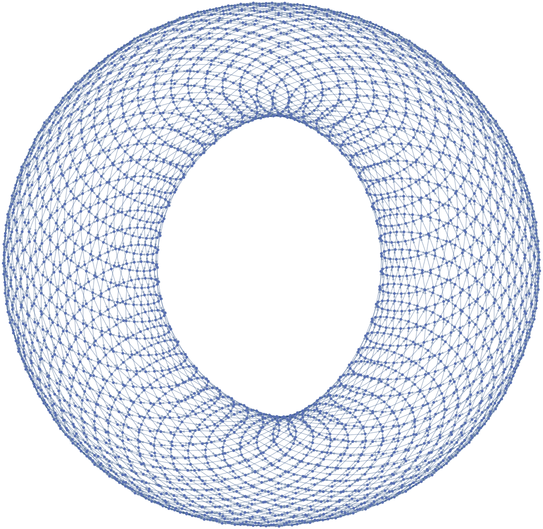
$$D(7) = \{0, 2, 3, 4, 6\}$$

Mod 23^2



$$D(23) = \{0, 7, 11, 15, 22\}$$

Mod 59^2



$$D(59) = \{0, 6, 29, 52, 58\}$$

Gessel's mod p^2 congruence

Define the sequence $A'(n)_{n\geq 0}$ by

$$A'(n) := 2\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 (H_{n+k} - H_{n-k}),$$

where $H_m := 1 + \frac{1}{2} + \cdots + \frac{1}{m}$ is the mth harmonic number.

$A'(n)_{n\geq 0}$: $0, 12, 210, 4438, 104825, \frac{13276637}{5}, 70543291, \frac{67890874657}{35}, \ldots$

Gessel notes: If $A(n)$ can be extended to a differentiable function $A(x)$ satisfying the same recurrence as $A(n)$, then $A'(n) = \left(\frac{d}{dx}A(x)\right)|_{x=n}$.

Theorem (Gessel 1982)

Let p be a prime. For all $d \in \{0, 1, \ldots, p-1\}$ and for all $n \geq 0$,

$$A(d + pn) \equiv \big(A(d) + pnA'(d)\big)A(n) \mod p^2.$$

Interpolation to $\mathbb{C}$

$A(n)$ is a hypergeometric function:

$$A(n) = \sum_{k \geq 0} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k \geq 0} \frac{\Gamma(n+k+1)^2}{\Gamma(n-k+1)^2 k!^4}$$
$$= \sum_{k \geq 0} \frac{(-n)_k (-n)_k (n+1)_k (n+1)_k}{k!^4}$$
$$= {}_4F_3(-n, -n, n+1, n+1; 1, 1, 1; 1)$$

where $(a)_k = a(a+1) \cdots (a+k-1)$ is the Pochhammer symbol.

For all $z \in \mathbb{C}$, define

$$A(z) := {}_4F_3(-z, -z, z+1, z+1; 1, 1, 1; 1).$$

Theorem (Rowland–Yassawi)

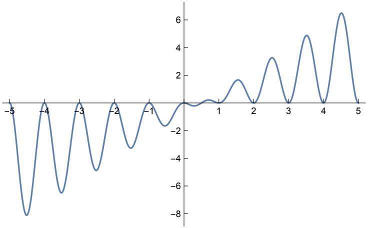
$A(z)$ is analytic at every $z \in \mathbb{C}$.

Recurrence

$\color{red}{R(z)} := z^3A(z) - (34z^3 - 51z^2 + 27z - 5)A(z-1) + (z-1)^3A(z-2)$

Then $R(n) = 0$ for every integer $n \geq 2$.

Plot of $R(z)$:



For all $z \in \mathbb{C}$,

$$R(z) = \frac{8}{\pi^2}(2z-1)(\sin(\pi z))^2.$$

This is sufficient, since $R'(z) = 0$ at integers $z = n$.

Series expansion

Power series:

$$A(z) = 1 + 0z + \zeta(2)z^2 + 2\zeta(3)z^3 - \frac{1}{2}\zeta(4)z^4 + \cdots$$

More terms of the power series:

$$A(z) = 1 + 0z + \zeta(2)z^2 + 2\zeta(3)z^3 - \frac{1}{2}\zeta(4)z^4 \\ - 4\zeta(3,2)z^5 + (2\zeta(2,4) - \zeta(4,2))z^6 + \cdots$$

The multiple zeta function is

$$\zeta(s_1, s_2, \ldots, s_m) := \sum_{n_1 > n_2 > \cdots > n_m > 0} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_m^{s_m}}.$$

$\zeta(3,2) = 3\zeta(2)\zeta(3) - \frac{11}{2}\zeta(5)$

References

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